

NIGEL KALTON AND THE INTERPOLATION THEORY OF COMMUTATORS

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ABSTRACT. This is the second of a series of papers surveying some small part of the remarkable work of our friend and colleague Nigel Kalton. We have written it as part of a tribute to his memory. It does not contain new results. One of the many topics in which Nigel made very significant and profound contributions deals with commutators in interpolation theory. It was our great privilege to work with him on one of his many papers about this topic. Our main purpose here is to offer an introduction to that paper: A unified theory of commutator estimates for a class of interpolation methods. *Adv. Math.* 169 (2002), no. 2, 241–312. We sketch the theory of interpolation spaces constructed using pseudolattices which was developed in that paper and which enables quite general formulation of commutator theorems. We seek to place the results of that paper in the general context of preceding and subsequent research on this topic, also indicating some applications to other fields of analysis and possible directions for future research.

1. INTRODUCTION

Given an initial pair of spaces, the classical interpolation theories provide us with general methods to construct new spaces that have the *interpolation property*, meaning that operators which are bounded on both the initial spaces are also bounded on the newly constructed spaces. A salient feature of these constructions is their generality: They apply equally well to all the operators that satisfy the prescribed initial assumptions. However, certain important operators in Analysis (some of which are nonlinear) can be shown to be bounded only because of subtle cancellations. For many of these operators the direct application of standard interpolation methods turned out to be inadequate, and the challenge was how to extend the classical theory to cope with this situation.

Although much more still needs to be done, in the last 30 years, through the work of a number of mathematicians, among whom Nigel Kalton was one of the leaders, we now have an extensive body of knowledge on how to handle commutators, including certain nonlinear ones. Moreover, the theory has naturally expanded to provide a general framework to study certain types of cancellations. Many new applications in disparate fields have been developed using these ideas.

Informally, the operators that originated the interpolation theory of commutators are of the form:

$$[T, \Omega] = T\Omega - \Omega T,$$

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where T is a bounded linear operator (a “good” operator) in some interpolation scale and Ω is a possibly nonlinear operator which is unbounded on relevant space(s) of the interpolation scale (a “bad” operator). At first sight it would seem that forming such a commutator is going to worsen our predicament, since it will require us to cope with two unbounded operators instead of one...but, somehow, the nasty behaviors of these two bad terms cancel each other out. This saves the day, and provides a bounded operator. Informally, we have the following scheme.

$$Bad - Bad = Good!$$

As already hinted above, phenomena of cancellation can be quite subtle and difficult to analyze. But it turns out to be possible, in most of the instances that we will describe here, to handle them via appropriately interpreted versions of the following simple fact: If two holomorphic functions have a simple pole at the same point and their residues at that point are equal, then their difference is holomorphic at that point.

In the next section of this document we will deal mostly with the background and some of the motivations for the theory presented in [18]. There is considerable overlap with the introductory section of [18] itself. Here we have allowed ourselves the luxury of presenting some of the examples in more detail, and adding a few more comments, some of them relating to some more recent developments.

Subsequently, in Section 3, we shall attempt to give the flavor of the results of [18]. Together with Nigel, we wrote [18] with the intention of making it applicable to as wide a range of settings as possible. This means that the reader must wade through several levels of definitions and conditions and extra remarks about better constants and weaker hypotheses. We hope that the simplified overview given in Section 3 will provide perspectives to make such a task somewhat easier.

Finally, in Section 4, we will suggest some possible directions for future research.

2. INTERPOLATION THEORY OF COMMUTATORS

While the main focus of interpolation theory is the systematic construction of spaces “with the interpolation property” from given ones (“interpolation methods”), it was observed in [41] that associated with these methods there are certain (unbounded and possibly nonlinear) operators Ω whose commutators with bounded operators can be controlled inside the corresponding interpolation scales. In due course it was realized that the resulting theory could be expanded to provide a general framework to analyze and control the underlying cancellations in different contexts.

2.1. Commutators in the setting of the complex method of interpolation.

Example 2.1. (The Coifman-Rochberg-Weiss Commutator Theorem [14]) It is well known that the space $BMO = BMO(\mathbb{R}^n)$ contains unbounded functions, in particular, if b is an element of $BMO \setminus L^\infty$, the multiplier operator $M_b(f) = bf$ is not bounded on, say, $L^p = L^p(\mathbb{R}^n)$. On the other hand, Calderón-Zygmund singular integral operators are known to be bounded on L^p , $1 < p < \infty$. The commutator theorem of Coifman-Rochberg-Weiss [14] asserts that the commutator between a Calderón-Zygmund operator T , and multiplication by a BMO function b , is a bounded operator on L^p :

$$(2.1) \quad [T, M_b] : L^p \rightarrow L^p.$$

In [14] one of the ways in which the authors proved the boundedness of the commutator $[T, M_b]$ on L^p was by identifying it (cf. [14, pp. 620–621]) as the derivative of a (Banach space valued) analytic function. They also used some consequences of a connection (formulated below in Example 2.3) of BMO with the A_p classes of weights.

Remark 2.2. We refer to [13, pp. 257–258] for the first of several applications in that paper of (2.1). It yields results about the properties of Jacobians of functions with prescribed smoothness. Related material can also be found in Chapter 13 of [27].

In [41], Rochberg and Weiss took the ideas of [14] one step further and, in the process, initiated the general interpolation theory of commutators. In particular, they introduced a class of operators which act on (and are associated with the construction of) individual complex interpolation spaces (cf. [6]). Although operators Ω in this class can be nonlinear and/or unbounded on their designated complex interpolation space, their commutators $[T, \Omega]$ are always bounded operators on that same interpolation space, for every choice of linear operator T which acts boundedly on both of the “endpoint” spaces.

The reader who is not yet familiar with Alberto Calderón’s complex method of interpolation can of course refer to [6] or, for example, to Chapter 4 of [4], or can perhaps make do with a brief overview, such as the one given in our earlier paper [19] in this series. Given a Banach pair $\vec{A} = (A_0, A_1)$, we consider the scale of complex interpolation spaces $A_t = [A_0, A_1]_t$, $t \in (0, 1)$. We fix a value of t and, for each $x \in A_t$, let $f_{x,t}(z)$ be its representing function, i.e. $f_{x,t} \in \mathcal{F}(\vec{A})$ is such that¹

$$f_{x,t}(t) = x, \text{ and } \|f_{x,t}\|_{\mathcal{F}} = \|x\|_{A_t}.$$

Then we let $\Omega_{\vec{A}} : A_t \rightarrow A_0 + A_1$, be defined by

$$(2.2) \quad \Omega_{\vec{A}} x = f'_{x,t}(t).$$

The abstract interpolation theorem of Rochberg-Weiss now reads: if \vec{A} and \vec{B} are Banach pairs and $T : \vec{A} \rightarrow \vec{B}$ is any bounded operator acting between them, then for each $t \in (0, 1)$ there exists a constant $c(t) > 0$ such that

$$(2.3) \quad \|[T, \Omega] x\|_{B_t} \leq c(t) \|T\|_{\vec{A} \rightarrow \vec{B}} \|x\|_{A_t},$$

where² $[T, \Omega] = T\Omega_{\vec{A}} - \Omega_{\vec{B}}T$. (One should bear in mind that $\Omega_{\vec{A}}$ and $\Omega_{\vec{B}}$ depend on our choice of t .)

The idea of the proof is to exploit the cancellation that is exhibited by the commutator $[T, \Omega]$. Given $x \in A_t$, let $G(z) = (T(f_{x,t}(z)) - f_{Tx,t}(z))/(z - t)$, then the cancellation $Tf_{x,t}(t) - f_{Tx,t}(t) = Tx - Tx = 0$, makes this function analytic also at $z = t$ with

$$(2.4) \quad G(t) = \lim_{z \rightarrow t} G(z) = \lim_{z \rightarrow t} \frac{T(f_{x,t}(z)) - f_{Tx,t}(z)}{z - t} = T\Omega_{\vec{A}} x - \Omega_{\vec{B}} Tx.$$

¹As in [19], in this discussion we have chosen to ignore issues concerning the existence and uniqueness of $f_{x,t}$. In the literature this difficulty is sometimes dealt with by working modulo bounded operators.

²During a first reading, you might prefer to just consider the special case where $\vec{A} = \vec{B}$ to avoid the slight inconvenience of having to deal with two different versions of Ω . In fact, via a straightforward use of direct sums, the general case can be deduced from this special case.

The desired result (2.3) will now follow immediately, once we show that $G \in \mathcal{F}(\vec{B})$. This is rather obvious, but for later purposes we will be more explicit: We have obtained G from a function in $\mathcal{F}(\vec{B})$ by multiplying that function by the scalar function $1/(z - t)$. We have seen that the singularity of that scalar function at $z = t$ does not create any problems. So the result follows trivially from the facts that $1/(z - t)$ is analytic everywhere else in the strip $\{z \in \mathbb{C} : 0 < \operatorname{Re} z < 1\}$ and is continuous and bounded on its closure.

Example 2.3. Let $p \in [1, \infty)$ and let w_0 and w_1 be positive functions on \mathbb{R}^n . Consider the pair of weighted L^p spaces³ $\vec{L}^p = (L^p(w_0), L^p(w_1))$. It is well known and easy to show that the complex method of interpolation gives

$$(2.5) \quad [L^p(w_0), L^p(w_1)]_t = L^p(w_0^{1-t}, w_1^t) \text{ for each } t \in (0, 1).$$

A simple calculation, which is in fact also a main step of the proof of (2.5), shows that, for each $t \in (0, 1)$ and for all elements x in a dense subspace of $[L^p(w_0), L^p(w_1)]_t$, we have $f_{x,t}(z) = (w_1/w_0)^{(z-t)/p}x$. So, in this case, for these elements⁴, the operator $\Omega_{\vec{L}^p}$ (cf. (2.2)) is given by

$$(2.6) \quad \Omega_{\vec{L}^p} = \frac{1}{p}x \log(w_1/w_0).$$

(Various versions of this calculation, in this and other more elaborate similar settings can be found in [41] pp. 335–340.)

This is an important special case of Example 2.3. If we now require $p > 1$ and if b is a BMO function, then, as more or less informally observed and/or used in a number of papers, including [12], [14], [23] and [41], etc.⁵, and explained in detail, e.g., in [21] p. 409, there exists $0 < \alpha = O(1/\|b\|_{BMO})$ (depending on p) such that $e^{-\alpha b}$ and $e^{\alpha b}$ both belong to the A_p class of Muckenhoupt weights. Therefore, if T is a Calderón-Zygmund singular integral operator, then $T : L^p(e^{-\alpha b}) \rightarrow L^p(e^{-\alpha b})$ and $T : L^p(e^{\alpha b}) \rightarrow L^p(e^{\alpha b})$ continuously. We have (cf. (2.5)) that $[L(e^{-\alpha b}), L(e^{\alpha b})]_{1/2} = L^p$ and so, by (2.6),

$$\Omega f = \frac{1}{p}f \log(e^{\alpha b}/e^{-\alpha b}) = \frac{2}{p}f \alpha b = \frac{2}{p}\alpha M_b(f).$$

Consequently, (2.3) with $\vec{A} = \vec{B} = \vec{L}^p$ gives

$$\|[T, M_p]f\|_{L^p} = \frac{p}{2\alpha} \|[T, \Omega]f\|_{L^p} \leq c \|b\|_{BMO} \|f\|_{L^p},$$

where the constant c depends on p and also on the norms of T as an operator on $L^p(e^{-\alpha b})$ and $L^p(e^{\alpha b})$. Thus we recover the commutator theorem of [14].

³In our discussion here, $L^p(w)$ on some measure space (X, Σ, μ) (usually \mathbb{R}^n with Lebesgue measure) is defined by requiring the norm $\|f\| = (\int_X |f|^p w d\mu)^{1/p}$ to be finite. But it should be noted that there are many papers in which it is found to be more convenient to replace w by w^p in this definition.

⁴There are also other examples where it can happen that some general explicit formula obtained for Ωx can be seen to coincide with $f'_{z,t}$ only for x in some dense subspace of the relevant space A_t . However this density may enable (2.3), with Ω given by that explicit formula, to be deduced for all $x \in A_t$. We are glossing over this issue here.

⁵In the case of functions of one variable this fact is linked to a much earlier theorem of Helson-Szegő. The authors of [12] also acknowledge contributions of R. Gundy and of the authors of [23].

Example 2.4. For the pair (L^1, L^∞) we have $[L^1, L^\infty]_t = L^p$ where $p = 1/(1-t)$ for each $t \in (0, 1)$. Then, for each simple function $x \in [L^1, L^\infty]_t$ we have

$$f_{x,t}(z) = \|x\|_p \frac{x}{|x|} \left(\frac{|x|}{\|x\|_p} \right)^{p(1-z)}$$

and, consequently,

$$\Omega x = -px \log \left(\frac{|x|}{\|x\|_p} \right).$$

See Proposition 1.5 on p. 166 of [20] for a detailed discussion of a non-commutative version of this example. Cf. also a related calculation, with some applications, on pp. 315–318 of [41].

2.2. Translation operators and commutators, and their application to partial differential equations. When studying operators on a scale of spaces it is also of interest to consider “translation” operators and their corresponding commutators. It should be made clear at the outset that here we are *not* talking about translation operators on function spaces, i.e., not about maps of the kind $f(\cdot) \mapsto f(\cdot - x_0)$ for some fixed x_0 in the underlying set. Instead it is the parameter of interpolation t which is being translated.

For the particular case of L^p spaces this idea was developed by Iwaniec [25] and Iwaniec and Sbordonc [28] (cf. Example 2.5 below) who found remarkable applications to nonlinear PDEs (cf. [25], [26], and the references therein). More generally, for the complex method of interpolation, these translation operators can be defined as follows. Suppose that $x \in A_t$ is represented by

$$x = f_{x,t}(t), \text{ with } \|x\|_{A_t} = \|f_{x,t}\|_{\mathcal{F}}.$$

Then, for sufficiently small $\varepsilon \in \mathbb{R}$, we define

$$\mathcal{R}_\varepsilon x = f_{x,t}(t + \varepsilon).$$

It follows that

$$\mathcal{R}_\varepsilon : A_t \rightarrow A_{t+\varepsilon}.$$

The general version of the commutator theorem for translations (cf. [18]) asserts that if $T : \vec{A} \rightarrow \vec{B}$ is a bounded linear operator, then for all $t \in (0, 1)$, there exists a constant $c = c(t) > 0$ such that⁶

$$(2.7) \quad \|\mathcal{R}_\varepsilon T x - T \mathcal{R}_\varepsilon x\|_{B_{t+\varepsilon}} \leq c |\varepsilon| \|T\|_{\vec{A} \rightarrow \vec{B}} \|x\|_{A_t},$$

where the constant c depends only on t .

Of course, for each $x \in A_t$, we have $\lim_{\varepsilon \rightarrow 0} (\mathcal{R}_\varepsilon x - x)/\varepsilon = \Omega x$ in suitable topologies (such as the norm topology in $A_0 + A_1$) which can lead us to think of (2.3) as a sort of limiting case⁷ of (2.7). In fact quite generalized versions of (2.3) and (2.7) (both formulated in Theorem 3.8 on page 256 of [18]) emerge easily and simultaneously from the same simple

⁶Here, analogously to what happens in (2.3), \mathcal{R}_ε has two different meanings in (2.7), depending on whether it acts on A_t or on B_t . Here again one can readily deduce the general case of this result from the special and more easily formulated case where $\vec{A} = \vec{B}$.

⁷This idea is brought into play in a rather more general setting in Section 4 of [35].

argument appearing in the lower part of page 259 of [18], once the required machinery has been set up.

Example 2.5. (A commutator theorem of Iwaniec-Sbordone [28]) We consider “translation operators” in the setting of L^p spaces. Let $1 \leq p_0 < p_1 \leq \infty$, be fixed, and let $\theta \in (0, 1)$. Let $p \in (p_0, p_1)$ be given by $1/p = (1 - \theta)/p_0 + \theta/p_1$ so that $L^p = [L^{p_0}, L^{p_1}]_\theta$. Moreover, for z on the strip $0 \leq \operatorname{Re} z \leq 1$, we let $h(z) = (1 - z)/p_0 + z/p_1$. Then, if f is a non-zero element of L^p , it can be represented in an optimal way, for the computation of its norm in the interpolation space L^p , by the analytic function

$$F_{f,\theta}(z) = \|f\|_p \left(\frac{|f|}{\|f\|_p} \right)^{h(z)p} \frac{f}{|f|}, \text{ with } 0 \leq \operatorname{Re} z \leq 1.$$

(The formula given above in Example 2.4 is a limiting case of this formula.) It follows that

$$\begin{aligned} \mathcal{R}_\varepsilon f &= F_{f,\theta}(\theta + \varepsilon) \\ &= \|f\|_p \left(\frac{|f|}{\|f\|_p} \right)^{h(\theta+\varepsilon)p} \frac{f}{|f|} \\ &= \left(\frac{|f|}{\|f\|_p} \right)^{\varepsilon p(\frac{1}{p_1} - \frac{1}{p_0})} f. \end{aligned}$$

Moreover,

$$\mathcal{R}_\varepsilon : L^p = [L^{p_0}, L^{p_1}]_\theta \rightarrow [L^{p_0}, L^{p_1}]_{\theta+\varepsilon} = L^{p_\varepsilon},$$

where

$$\frac{1}{p_\varepsilon} = \frac{1 - (\theta + \varepsilon)}{p_0} + \frac{(\theta + \varepsilon)}{p_1} = \frac{1}{p} + \varepsilon \left(\frac{1}{p_1} - \frac{1}{p_0} \right).$$

Then, the following commutator theorem, which is essentially the same as Theorem 4 of [28, p. 147], can be seen to also follow from (2.7): *Let T be a bounded operator on L^{p_j} for $j = 0, 1$. Then for each ε such that $0 < \theta + \varepsilon < 1$ and for*

$$[\mathcal{R}_\varepsilon, T]f = \mathcal{R}_\varepsilon(Tf) - T(\mathcal{R}_\varepsilon f) = \left(\frac{|Tf|}{\|Tf\|_p} \right)^{\varepsilon p(\frac{1}{p_1} - \frac{1}{p_0})} (Tf) - T \left(\left(\frac{|f|}{\|f\|_p} \right)^{\varepsilon p(\frac{1}{p_1} - \frac{1}{p_0})} f \right)$$

we have

$$\|[\mathcal{R}_\varepsilon, T]f\|_{L^{p_\varepsilon}} \leq c |\varepsilon| \|f\|_{L^p},$$

where c depends only on $p, p_0, p_1, \|T\|_{L^{p_0} \rightarrow L^{p_0}}$ and $\|T\|_{L^{p_1} \rightarrow L^{p_1}}$.

Remark 2.6. Theorem 4 of [28] plays an important role in that paper. It is a tool there for obtaining results concerning existence and regularity properties of solutions (including vector valued ones) of the Dirichlet problem on certain domains $W \subset \mathbb{R}^n$, of a large class of divergence type partial differential equations. That class includes the familiar p -harmonic equation $\operatorname{div}(|\nabla u|^{p-2} \nabla u) = 0$. In this application the role of T is played by a certain singular integral operator \mathcal{K}_W associated with the so-called Hodge decomposition of vector fields defined on W . On page 147 of [28] there is also a brief indication of other possible significant applications of Theorem 4 of that paper to nonlinear potential theory and other

topics. Theorem 13.2.1 on p. 341 of the book [27] is also essentially the same as the theorem formulated just above, and is presented there with rather different applications in mind. More generally, Chapter 13 of [27] offers a number of applications of the ideas we have been discussing to problems in analysis.

2.3. Furthermore. In [31] (see also our introduction to [31] in [19]) Nigel defined *a priori* a class of operators Ω that form bounded commutators with bounded operators on a large class of Köthe spaces. Moreover, in [32], he and Marius Mitrea considered a very general interpolation method for quasi-Banach spaces, which contains both complex interpolation and real interpolation as special cases, and has an in-built cancellation principle (expressed by Property (3) on p. 3905 of [32]). Their applications included a study of the stability of isomorphism or Fredholm properties etc. of operators on complex interpolation scales of quasi-Banach spaces. By “stability” we mean results of the following kind: Suppose that the spaces A_t form an interpolation scale with respect to the complex (or some other) method, and that the linear operator T is bounded on each A_t and has some additional property (such as being a Fredholm operator) on A_t when t takes some particular value $t = t_0$. Then there exists some open neighbourhood of t_0 such that T has that same additional property on A_t for all t in that neighbourhood.

2.4. Methods of real interpolation, and higher order commutators. The ideas of [41] were extended to the real method of interpolation by Jawerth-Rochberg-Weiss [30]. In that paper, suitable operators Ω were defined for the K , J and E methods of interpolation, and corresponding new commutator theorems were obtained for them. Explicit calculations of these Ω operators showed that they depend on the particular method of interpolation used. It was also shown that for the J -method one can provide a treatment of the theory of commutators that closely follows the analysis of the complex method. The form and effects of cancellations for the real method of interpolation were discussed in some detail in [38].

Another natural issue that arose at the time was the problem of dealing with higher order commutators. Simple iteration does not work since the (first order) commutator $[T, \Omega]$ is, in general, not linear. Nevertheless, higher order versions of the commutator theorems for the complex method were obtained in [42] and the corresponding results for the real methods followed suit (cf. [43] for a recent survey).

3. A UNIFIED METHOD OF INTERPOLATION

Now we come to the review of the paper [18] proper. As the interpolation theory of commutators was developing, many of the results had to be derived, one method of interpolation at a time, and via different arguments tailored for each specific method of interpolation. The treatment of the complex method was particularly elegant and amenable to the study of cancellations, higher order inequalities, and further applications. However, the study of commutators in the contexts of other methods of interpolation was somewhat more laborious.

This prompted the following question: Might it be possible to find some unified way of describing several different interpolation methods, in a manner which also provides a unified way of obtaining diverse known (and hopefully also new) commutator results for

those different methods? Our work [18] in 2002 with Nigel⁸ provided an affirmative answer to this question (and also showed that the known straightforward approach for the complex method could be adapted to work for other methods). But ours was not the first such answer.

In 1981 the impressive work of Svante Janson [29] had shown that most known interpolation methods fit into the general Aronszajn-Gagliardo framework of orbital and co-orbital methods (see also [39]). But at that time there did not seem to be any way of dealing with commutators in a context as general as this. (See however item 4 in Section 4 regarding some progress decades later towards such a goal.)

In 1995 Maria Carro, Joan Cerdà and Javier Soria [8], using a construction due to Vernon Williams [47] as their starting point, developed a quite general interpolation theory which did enable treatment of commutators, and in which known commutator theorems, for example, for the complex method and for the real J and K methods all readily appeared as special cases. They gave further discussion and development of their work in [9, 11, 10]. In several places in our paper, in particular in Section 5 on pp. 276–279, the reader can find detailed comparisons of their approach and ours to this topic.

Our paper was dedicated to Jaak Peetre on the occasion of his 65th birthday, (and the dedication even appears in his native Estonian⁹). We felt this to be particularly appropriate, since Jaak’s 1971 paper [40] had essentially shown us the way to give several other interpolation methods a holomorphic “structure” similar, one might even say surprisingly similar, to that which appears intrinsically in the complex method. Let us indicate how this happened: In [40], among many other things, Jaak introduced a new method of interpolation, sometimes since called the \pm method. The spaces which it generates are usually denoted by $\langle A_0, A_1 \rangle_\theta$. Jaak observed that they are related to real and complex interpolation spaces by the inclusions $(A_0, A_1)_{\theta,1} \subset \langle A_0, A_1 \rangle_\theta \subset [A_0, A_1]_\theta$. In fact in his discussion of these matters he worked with a certain variant of the complex method, about which we will now need to say a few words. We will temporarily use the notation $[A_0, A_1]_\theta^{(\mathbb{A})}$ for the spaces obtained by this different method. Let $\mathbb{S} = \{z \in \mathbb{C} : 0 < \operatorname{Re} z < 1\}$ and $\mathbb{A} = \{z \in \mathbb{C} : 1 < |z| < e\}$. It is very easy to see that the definition of the spaces $[A_0, A_1]_\theta^{(\mathbb{A})}$ as formulated in [40] can be equivalently reformulated to be the same as Calderón’s definition in [6], except that the closed strip $\overline{\mathbb{S}} = \{z \in \mathbb{C} : 0 \leq \operatorname{Re} z \leq 1\}$, which was used by Calderón in [6] (and originally by Thorin), must be replaced in a rather obvious way, by a closed annulus, say¹⁰ $\overline{\mathbb{A}} = \{z \in \mathbb{C} : 1 \leq |z| \leq e\}$. More explicitly, let $\mathcal{F}_{\mathbb{A}}(A_0, A_1)$ be the space of continuous functions $f : \overline{\mathbb{A}} \rightarrow A_0 + A_1$ which are holomorphic in \mathbb{A} and, for $j = 0, 1$, their restrictions to the circles $|z| = e^j$ are continuous A_j -valued functions. Let us norm $\mathcal{F}_{\mathbb{A}}(A_0, A_1)$ by

$$\|f\|_{\mathcal{F}_{\mathbb{A}}(A_0, A_1)} = \max_{j=0,1} \sup \left\{ \|f(e^{j+it})\|_{A_j} : 0 \leq t \leq 2\pi \right\}.$$

⁸Throughout the rest of this document, please understand “we”, “our” or “ours” to often be referring to *all* authors of [18], of course including Nigel.

⁹It was kindly translated into Estonian for us by the late Simson Baron.

¹⁰To make our definition here correspond *exactly* to the definition in [40] the outer radius of our annulus should be 2 instead of e . But (see the next footnote) the choice of radius does not really matter.

Now we can define $[A_0, A_1]_\theta^{(\mathbb{A})}$ to be the space of all elements $a \in A_0 + A_1$ of the form $a = f(e^\theta)$ for some $f \in \mathcal{F}_{\mathbb{A}}(A_0, A_1)$. The norm in $[A_0, A_1]_\theta^{(\mathbb{A})}$ is defined exactly analogously to Calderón's definition of the norm in $[A_0, A_1]_\theta$, i.e., by

$$(3.1) \quad \|a\|_{[A_0, A_1]_\theta^{(\mathbb{A})}} = \inf \left\{ \|f\|_{\mathcal{F}_{\mathbb{A}}(A_0, A_1)} : f \in \mathcal{F}_{\mathbb{A}}(A_0, A_1), a = f(e^\theta) \right\}.$$

In [40] and elsewhere, Jaak Peetre asked whether $[A_0, A_1]_\theta^{(\mathbb{A})}$ is the same space as $[A_0, A_1]_\theta$. The answer [15] turned out to be yes, they are equal¹¹, to within equivalence of norms.

Now let us rewrite the above definition of $[A_0, A_1]_\theta^{(\mathbb{A})}$ in terms of the coefficients which appear in the Laurent expansions $f(z) = \sum_{n \in \mathbb{Z}} z^n a_n$ in $\overline{\mathbb{A}}$ of functions $f \in \mathcal{F}_{\mathbb{A}}(A_0, A_1)$. It is easy to show that the coefficients a_n in such expansions must all be elements of $A_0 \cap A_1$. Our equivalent definition has to be as follows:

Definition 3.1. The space $[A_0, A_1]_\theta^{(\mathbb{A})}$ consists of all elements $a \in A_0 + A_1$ which can be written in the form

$$(3.2) \quad a = \sum_{n \in \mathbb{Z}} e^{\theta n} a_n$$

where $\{a_n\}_{n \in \mathbb{Z}}$ is a sequence of elements in $A_0 \cap A_1$ such that, for $j = 0$ and $j = 1$, the “weighted” sequence $\{e^{jn} a_n\}_{n \in \mathbb{Z}}$ belongs to the sequence space $FC(A_j)$.

Here, for any Banach space¹² B , the sequence space $FC(B)$ is defined to consist of all B -valued sequences $\{b_n\}_{n \in \mathbb{Z}}$ which arise as the Fourier coefficients of a (uniquely determined) continuous B -valued function $\phi : \mathbb{T} \rightarrow B$, and the norm $\|\{b_n\}_{n \in \mathbb{Z}}\|_{FC(B)}$ of such a sequence equals $\sup \{\|\phi(e^{it})\|_B : t \in [0, 2\pi]\}$.

It is convenient to let $\mathcal{J}(FC(A_0), FC(A_1))$ denote the space of all sequences $\{a_n\}_{n \in \mathbb{Z}}$ which satisfy the two conditions specified in Definition 3.1 and to equip it with the norm

$$(3.3) \quad \|\{a_n\}_{n \in \mathbb{Z}}\|_{\mathcal{J}(FC(A_0), FC(A_1))} = \max_{j=0,1} \left\| \{e^{jn} a_n\}_{n \in \mathbb{Z}} \right\|_{FC(A_j)}.$$

So $\mathcal{J}(FC(A_0), FC(A_1))$ is simply the “Fourier transform” of $\mathcal{F}_{\mathbb{A}}(A_0, A_1)$ and we can rewrite (3.1) as

$$(3.4) \quad \|a\|_{[A_0, A_1]_\theta^{(\mathbb{A})}} = \inf \left\{ \|\{a_n\}_{n \in \mathbb{Z}}\|_{\mathcal{J}(FC(A_0), FC(A_1))} \right\}$$

where the infimum is taken over all sequences $\{a_n\}_{n \in \mathbb{Z}}$ in $\mathcal{J}(FC(A_0), FC(A_1))$ which satisfy (3.2).

The definition of $[A_0, A_1]_\theta^{(\mathbb{A})}$, when expressed in the language of Definition 3.1, looks very much like one of the equivalent definitions of the Lions-Peetre real interpolation space which is usually denoted by $(A_0, A_1)_{\theta, p}$. (Cf. e.g. [4, Lemma 3.2.3 p. 43] or [36, p. 17], where different notation is used for this space.) If we simply replace the spaces $FC(A_0)$ and

¹¹The same result holds when the annulus defining this space is replaced by $\overline{A(r)} := \{z : 1 \leq |z| \leq r\}$ for any choice of $r > 1$. The norm of the complex interpolation space defined using $\overline{A(r)}$ is only known to be equivalent to the norm of the space $[A_0, A_1]_\theta$, and it is known that in general these norms cannot be equal, e.g. for large values of r . Recently Eliran Avni [1] has shown that the norm of this space becomes arbitrarily close to the norm of $[A_0, A_1]_\theta$ as r tends to 1.

¹²i.e., any *complex* Banach space. All Banach spaces in this discussion are assumed to be complex.

$FC(A_1)$ respectively by $\ell^p(A_0)$ and $\ell^p(A_1)$ then it becomes exactly that definition. (Here, as usual, $\ell^p(B)$ is the space of all B -valued sequences $\{b_n\}_{n \in \mathbb{Z}}$ for which $(\sum_{n \in \mathbb{Z}} \|b_n\|_B^p)^{1/p}$ is finite.) Furthermore, replacing $FC(A_0)$ and $FC(A_1)$ by $\ell^p(A_0)$ and $\ell^p(A_1)$ in the definition of $\mathcal{J}(FC(A_0), FC(A_1))$ and then also in (3.4), converts (3.4) (of course together with the words immediately following it) into a formula for one of the equivalent norms of $(A_0, A_1)_{\theta, p}$.

Yet another analogous change will convert Definition 3.1 and the accompanying definition of the norm of $[A_0, A_1]_{\theta}^{(\mathbb{A})}$, into the definition of the space $\langle A_0, A_1 \rangle_{\theta}$ and its norm. This time $FC(A_0)$ and $FC(A_1)$ must be replaced by the sequence spaces $UC(A_0)$ and $UC(A_1)$ whose definitions can be found in [18, Example 2.4, p. 247]. The Gustavsson-Peetre variant of $\langle A_0, A_1 \rangle_{\theta}$ (introduced in [22]) will be obtained if we replace $FC(A_0)$ and $FC(A_1)$ by certain other sequence spaces ([18, Example 2.5, p. 247]).

Almost all of the above was explicit or implicit in [40]. The first step that we took in [18] (in Section 2 of that paper) was to incorporate all of the interpolation spaces given by the above definitions, and also other more exotic ones, into the general framework of *interpolation spaces defined via pseudolattices*. We defined a pseudolattice to be a map (or one might say functor) \mathcal{X} which, when applied to any complex Banach space B , yields a certain sequence space $\mathcal{X}(B)$ of B -valued sequences. This map was required to satisfy some simple conditions which we will not list here. (See [18, Definition 2.1 on p. 246].) For example, if we choose some suitable Banach lattice X on \mathbb{Z} , whose elements can thus be considered as complex valued sequences $\{\alpha_n\}_{n \in \mathbb{Z}}$, then we can use X to define a particular pseudolattice \mathcal{X} by requiring that, for each Banach space B , the space $\mathcal{X}(B)$ is the space (often denoted by $X(B)$) of all sequences $\{b_n\}_{n \in \mathbb{Z}}$ in B for which $\{\|b_n\|_B\}_{n \in \mathbb{Z}} \in X$. This is exactly what we did above in the special case $X = \ell^p$. In fact our discussion just above implicitly defines three pseudolattices: ℓ^p and also FC and UC .

Our definition of pseudolattice was formulated so that each choice of pseudolattice \mathcal{X} and parameter $\theta \in (0, 1)$ would provide us with a new interpolation method. This method, when applied to any Banach pair (A_0, A_1) yields an interpolation space which will be denoted here¹³ by $(A_0, A_1)_{\mathcal{X}, \theta}$. As can be anticipated from the preceding discussion, its definition and its norm were obtained by simply replacing $FC(A_0)$ and $FC(A_1)$ by $\mathcal{X}(A_0)$ and $\mathcal{X}(A_1)$ throughout the preceding definitions.

In order to get to this stage, it had been convenient to make the transition from holomorphic functions $f : \mathbb{A} \rightarrow A_0 + A_1$, to $A_0 \cap A_1$ valued sequences $\{a_n\}_{n \in \mathbb{Z}}$ (the Laurent coefficients of f). But then we wanted to go from these sequences back to holomorphic functions on \mathbb{A} . It turned out that a certain mild condition (which we called *Laurent compatibility*¹⁴) on \mathcal{X} , (and which holds in all “natural” examples known so far) suffices to ensure that, for every sequence $\{a_n\}_{n \in \mathbb{Z}}$ in $\mathcal{J}(\mathcal{X}(A_0), \mathcal{X}(A_1))$, the series $\sum_{n \in \mathbb{Z}} z^n a_n$ converges for all $z \in \mathbb{A}$ to a holomorphic $A_0 + A_1$ -valued function.

¹³In order to simplify our presentation here we are using notation which is a little different from that of [18] and not considering the more general version of this definition there. In that version we had the additional options of applying *different* pseudolattices to A_0 and A_1 and of permitting the parameter θ to also have a non-zero imaginary part. I.e., there we replaced the parameter $\theta \in (0, 1)$ by the parameter $s = e^{\theta} \in \mathbb{A}$.

¹⁴The original definition of Laurent compatibility in [18] (see Definition 2.9 on p. 248) was formulated for *pairs* of pseudolattices.

Now things began to look good. (We are now talking about Section 3 of [18].) For each $\theta \in (0, 1)$ we expected to be able to define an operator Ω for $A_\theta = (A_0, A_1)_{\mathcal{X}, \theta}$ very similarly to how this had been done in (2.2) and then to prove a more general result like (2.3) by the same kind of simple argument that was sketched above just after (2.3). We also expected to be able to define an analogue of the operator \mathcal{R}_ε of Subsection 2.2 and obtain a result like (2.7) for it. Indeed (via our statement and proof of Theorem 3.8 of [18, p. 256]), we would accomplish both these tasks. But there were two issues to be dealt with before this could happen.

The first of these was that (as indeed in the previously considered special cases of such results) we could not be sure that in general the infimum in the generalized version of (3.4) is attained by some sequence. (In fact an analogous concern arises already for the prototype definition in (2.2).) Even if this infimum is attained, the sequence attaining it may fail to be unique. So how can we define Ωa or $\mathcal{R}_\varepsilon a$ for each $a \in (A_0, A_1)_{\mathcal{X}, \theta}$? We overcame or bypassed this difficulty rather easily, by allowing Ωa and $\mathcal{R}_\varepsilon a$ to be appropriate *sets* of elements, rather than single elements. As stated in [18, Definition 3.1, p. 253], for some chosen constant $C_{opt} > 1$, these sets are, respectively, the set of all values of $f'(e^\theta)$ or of $f(e^{\theta+\varepsilon})$ as f ranges over all holomorphic functions $f : \mathbb{A} \rightarrow A_0 + A_1$ which satisfy $f(e^\theta) = a$, and whose sequences of Laurent coefficients have norms in $\mathcal{J}(\mathcal{X}(A_0), \mathcal{X}(A_1))$ which are no greater than C_{opt} times the infimum defining the norm $\|a\|_{(A_0, A_1)_{\mathcal{X}, \theta}}$ of a . (Later below there will be some brief discussion of Ω_n , an “ n -order” version of Ω . For each a as above, $\Omega_n a$ is obtained by replacing $f'(e^\theta)$ by $f^{(n)}(e^\theta)$ in the above definition.)

To explain the second issue, one should look again at the above-mentioned proof of the special case sketched just after (2.3). Suppose that $g(z) = \sum_{n \in \mathbb{Z}} z^n a_n$ for some sequence $\{a_n\}_{n \in \mathbb{Z}}$ in $\mathcal{J}(\mathcal{X}(A_0), \mathcal{X}(A_1))$ and that $g(e^\theta) = 0$. Here, analogously to the last step (immediately after (2.4)) of that proof, we needed to know that, if we multiply $g(z)$ by the scalar function $1/(z - e^\theta)$, then we will obtain a new holomorphic function on \mathbb{A} whose Laurent coefficients keep the same property as that possessed by those of g , i.e., they too are a sequence in $\mathcal{J}(\mathcal{X}(A_0), \mathcal{X}(A_1))$. It turned out that there is a very simple condition which suffices to ensure that this happens, and this condition is satisfied when \mathcal{X} is FC or ℓ^p or UC or any one of a large number of other “natural” choices. I.e., it suffices if, for each Banach space B , the left shift operator $\{b_n\}_{n \in \mathbb{N}} \mapsto \{b_{n+1}\}_{n \in \mathbb{N}}$ maps the sequence space $\mathcal{X}(B)$ isometrically onto itself. (Then of course the right shift operator $\{b_n\}_{n \in \mathbb{N}} \mapsto \{b_{n-1}\}_{n \in \mathbb{N}}$ has the same property.)

In fact we could also manage without imposing this isometry of shifts condition on \mathcal{X} . It turned out that a much weaker but more technical condition¹⁵ on \mathcal{X} is also sufficient to ensure that multiplication of $g(z)$ by $1/(z - e^\theta)$ has the property that we require.

Now, having described the path towards it, and the conditions required to take that path, we can invite you to look at Theorem 3.8 of [18, p. 256]. As you can see, it includes results mentioned above as special cases. In particular, the inequalities (3.2) and (3.3) in the statement of that theorem indeed generalize (2.3) and (2.7), respectively. Furthermore, as we showed after proving it, (see Section 4 of [18]) a number of other previously known

¹⁵We used the terminology \mathcal{X} *admits differentiation* for this property. In fact (see Definition 3.4 of [18, pp. 254–255]) we defined and dealt with *pairs* of pseudolattices which admit differentiation. Cf. also Lemma 3.6 of [18] p. 255 for another condition which can work in place of isometry of shifts.

results for commutators, including some in the context of versions of the real interpolation method, can also be deduced from Theorem 3.8.

Remark 3.2. The applicability of special cases of Theorem 3.8 to various topics in analysis (cf. Remark 2.2 and the comments near the beginning of Subsection 2.2) suggest that the similar but more general perspectives offered by Theorem 3.8 should also have useful applications.

It remains to give some brief account of what can be found in the several subsequent sections of [18]. As already mentioned, Section 5 compared our approach with that of Carro-Cerdà-Soria. Then in Section 6 we turned to a study of higher order analogs of Theorem 3.8. The natural definition of Ω_n , the n th order analog of Ω has already been mentioned above. For bounded operators T , our first theorem in Section 6 did not control the norm of $[T, \Omega_n]a$ but rather of a more elaborate inductively defined expression which contains “correction terms” also involving Ω_k for $k = 1, 2, \dots, n - 1$. Analogs of this result had been obtained earlier in [9] and also, in the particular contexts of real or complex interpolation methods, in [42] and [37]. (Cf. also [7] and [38].) We also obtained a second theorem, this time for translation operators \mathcal{R}_ε , which generalized the pseudolattice version of the norm estimate (2.7). Here the factor $|\varepsilon|$ on the right side of the estimate could be replaced by $|\varepsilon|^n$. Analogously to the preceding theorem for Ω_n , in this theorem the commutator expression of T and \mathcal{R}_ε on the left side whose norm is controlled by this power of $|\varepsilon|$ here also has to be quite elaborate and include extra “correction terms” which also involve Ω_k for $k = 1, 2, \dots, n - 1$.

As indicated above (in particular in Remarks 2.2 and 2.6) nonlinear first order commutators have already found quite a range of applications. It seems reasonable to expect that higher order results, such as those that we have mentioned here, will also eventually find interesting applications in various branches of analysis. For example, perhaps they could turn out to be useful in the study of functional equations (cf. [34] and the references therein).

In Section 7 of [18] we extended results about domain spaces and range spaces of operators Ω previously known for the real and complex methods to the general pseudolattice method. In both this and the previous section we noted connections with the Lions-Schechter variant [44] of the complex interpolation spaces. (We showed later, in an appendix, that also for the definition of these spaces, one can replace the strip by an annulus.)

Finally, to describe the result of Section 8, we first need to recall that there are two main ways of constructing the Lions-Peetre real interpolation spaces $(A_0, A_1)_{\theta, p}$. These are usually referred to as the J -method and the K -method. The fact that these two methods give the same spaces is sometimes quite helpful, for example for characterizing the duals of these spaces or for proving reiteration formulae (e.g. for describing the space $\left((A_0, A_1)_{\theta_0, p_0}, (A_0, A_1)_{\theta_1, p_1}\right)_{\theta_3, p_3}$). The construction of the space $(A_0, A_1)_{\mathcal{X}, \theta}$ for an arbitrary pseudolattice \mathcal{X} is modeled on the J -method (which explains our choice of notation for the related space $\mathcal{J}(\mathcal{X}(A_0), \mathcal{X}(A_1))$). Somewhat to our surprise, it turned out to be possible to formulate an alternative way of constructing the general spaces $(A_0, A_1)_{\mathcal{X}, \theta}$ which is modeled on the K -method for $(A_0, A_1)_{\theta, p}$. We are not aware of this fact having been observed earlier, not even for the important and much studied special case of complex interpolation spaces

$[A_0, A_1]_\theta$. Results about duality and reiteration are well known for $[A_0, A_1]_\theta$. But we keep the hope that this unexpected K -method will prove to be useful for other purposes.

4. FURTHER DIRECTIONS FOR FUTURE RESEARCH

In this final section we offer some additional comments, and formulations of some natural open problems associated with results discussed in the preceding sections.

- (1) The first project asks for a version of the interpolation of analytic families of operators of Stein [45] (cf. also [17]), in the context of the interpolation spaces of [18]. (Cf. also the notion of “interpolating family of operators” introduced in [32, p. 3905].)
- (2) Other open problems for the interpolation method of [18] include: the duality theory, bilinear interpolation and the study of compactness for these methods. When dealing with these it may turn out to be helpful to know that a lemma of Stafney, which is sometimes useful for the study of the complex method has been shown [24] to also hold for this general method.
- (3) Much remains to be done in the theory of interpolation of commutators. In particular, the problem of compactness of commutators has not been treated in the context of the abstract theory. The classical result due to Uchiyama [46] which one could seek to generalize states that if T is a Calderón-Zygmund operator and $b \in VMO$, and $p \in (1, \infty)$, then the commutator $[T, M_b] : L^p \rightarrow L^p$ is compact. What are the compactness results in the general theory? The problem is wide open even for the classical methods of interpolation. (Note that Bényi and Torres [3] have generalized Uchiyama’s result [46] to the context of bilinear operators.)
- (4) We have seen that cancellation principles can be formulated and used to obtain commutator theorems in quite general contexts, as in [8], [18], [32], etc. But still greater generality is possible. An abstract cancellation principle for orbital interpolation methods generated by a single element appears in [35] and is applied, not only to commutators, but also to stability¹⁶ theorems. Such orbital methods can be seen to extend some of the classical methods. (See [29, 35]). But are there corresponding results for orbital methods with a finite number of generators or, better still, without any such restrictions? Likewise what is the corresponding theory of commutators for co-orbital methods? (The work of [35] has a sequel [33] which presents some higher order results.)
- (5) Establish bilinear commutator theorems. The problem seems to be open even for the classical methods of interpolation. As already mentioned in item (3), some specialized bilinear results can be found in the paper [3].
- (6) The commutator theorem of [14] involves the commutator of singular integrals and the multiplication operator M_b , with $b \in BMO$. The method of [41] also gives the L^p boundedness of commutators $[T, M_b]$ for all operators T that are bounded on $L^p(w)$ for all weights in the Muckenhoupt class A_p ¹⁷. More generally, the result can be extended to some nonlinear operators T that satisfy the same weighted norm inequalities. For example (cf. [2] and the references therein), if T is either the

¹⁶Here “stability” has the same meaning as specified in Subsection 2.3.

¹⁷This follows by the same reasoning as that used in Example 2.3 above.

maximal operator of Hardy-Littlewood or the sharp maximal operator, then $[T, M_b]$ is bounded on L^p if and only if $b \in BMO$ and its negative part b^- is bounded¹⁸. While some other results are known for quasilinear operators (cf. [16]) it would be of interest to have a fully developed systematic theory of nonlinear commutators.

- (7) The problems that we propose here are motivated by an interesting application of commutator theorems given in [5] in connection with the definition of the product operator M_b in H^1 . One learns of the important roles in concrete applications of Orlicz spaces as well as Lorentz and Marcinkiewicz spaces in the commutator theory of interpolation. These give us good reasons to consider the project of finding an extension of the theory of [18] to be able to incorporate such refinements.

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¹⁸In particular if $b \in BMO$ and $b \geq 0$.

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